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GLOBAL ASYMPTOTIC STABILITY OF COHEN–GROSSBERG NEURAL NETWORKS OF NEUTRAL TYPE

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1. Introduction

An artificial neural network is an information-processing paradigm inspired by the way of processing the data by biological nervous systems, e.g., by the brain. The key element of this paradigm is the novel structure of the information processing system. It is composed of a large number of highly interconnected processing elements (neurons) working in unison to solve specific problems. Although the initial intent of artificial neural networks was to explore and reproduce human information-processing tasks, such as speech, vision, and knowledge processing, artificial neural networks also demonstrated their superior capability for the classification and function approximation problems. This has a great potential for the solution of complex problems, such as systems control, data compression, optimization problems, pattern recognition, and system identification.

The Cohen–Grossberg neural network [10] and its various generalizations with or without transmission delays and impulsive state displacements have been a subject of intense recent investigations [3, 6, 7, 13, 16, 17]. In a model of Cohen–Grossberg neural network, the feedback terms consist of amplification and stabilizing functions. They are, in general, nonlinear. These terms give a model with a special kind of generalization containing numerous neural network models with content addressable memory, such as additive neural networks, cellular neural networks and bidirectional associative memory networks, and also biological models (e.g., the Lotka–Volterra models of population dynamics), as special cases.

Unlike retarded systems, in neutral systems, time delays explicitly appear in the state velocity vector. Neutral systems can be applied to describe more complicated nonlinear engineering and bioscience models, including the models describing chemical reactors, transmission lines, partial element equivalent circuits in very large-scale integrated systems, and Lotka–Volterra systems [18, 14, 4, 1, 2, 9, 15]. Neural networks can be implemented using very large-scale integrated circuits. Therefore, both retarded-type delays and neutral-type delays are inherent in the dynamics of neural networks.

In the present paper, we consider a Cohen–Grossberg neural network of neutral type more general than in [8]. A discrete-time analog of this system endowed with impulsive conditions was considered in our previous paper [11]. Sufficient conditions for the global asymptotic stability of unique equilibrium point of the system are obtained by using an appropriate Lyapunov functional. The established conditions are much more precise than in [8]. An example is given.

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2. Preliminaries

We consider a Cohen–Grossberg neural network of neutral type consisting of $m \geq 2$ elementary processing units (or neurons) whose variables of state x_i (in what follows, we write $i = \overline{1,m}$ instead of $i = 1, 2, \ldots, m$) are governed by the system

$$\dot{x}_i(t) + \sum_{j=1}^{m} e_{ij}\dot{x}_j(t-\tau_j) = a_i(x_i(t))\left[-b_i(x_i(t)) + \sum_{j=1}^{m} c_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} d_{ij} g_j(x_j(t-\tau_j)) + I_i\right], \tag{1}$$

$$i = \overline{1,m}, \quad t > t_0 = 0,$$

with initial values specified by continuous functions $x_i(s) = \phi_i(s)$ for $s \in [-\tau, 0]$, $\tau = \max_{j=\overline{1,m}}\{\tau_j\}$. In (1), $a_i(x_i)$ denotes an amplification function, $b_i(x_i)$ denotes an appropriate function supporting the stabilizing (or negative) feedback term $-a_i(x_i)b_i(x_i)$ of the unit i, $f_j(x_j)$ and $g_j(x_j)$ denote activation functions, the parameters c_{ij} and d_{ij} are real numbers representing the weights (or strengths) of the synaptic connections between the j th unit and the i th unit without and with time delays, respectively, τ_j, the real numbers e_{ij} show how the state velocities of the neurons are delay feed-forward connected in the network, and the real constant I_i represents an input signal introduced from the outside to the i th unit of the network.

Let E be the unit $(m \times m)$-matrix. By $\mathcal{E}$ and $|\mathcal{E}|$, we denote the $(m \times m)$-matrices with entries e_{ij} and $|e_{ij}|$, respectively.

Definition 1 [5]. *A real matrix $A = (a_{ij})_{m\times m}$ is said to be an M-matrix if $a_{ij} \leq 0$ for $i, j = \overline{1,m}$, $i \neq j$, and all successive principal minors of A are positive.*

The assumptions accompanying the neural network (1) can be formulated as follows:

A$_1$. The amplification functions $a_i : \mathbb{R} \to \mathbb{R}^+$ are continuous and bounded in a sense that

$$0 < \underline{a}_i \leq a_i(x) \leq \overline{a}_i \quad \text{for} \quad x \in \mathbb{R}, \quad i = \overline{1,m},$$

for some constants $\underline{a}_i$ and $\overline{a}_i$.

A$_2$. The stabilizing functions $b_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous and monotonically increasing, namely,

$$0 < \underline{b}_i \leq \frac{b_i(x) - b_i(y)}{x - y} \leq \overline{b}_i \quad \text{for} \quad x \neq y, \quad x, y \in \mathbb{R}, \quad i = \overline{1,m},$$

for some constants $\underline{b}_i$ and $\overline{b}_i$.

A$_3$. The activation functions $f_j, g_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous in a sense that

$$\sup_{x\neq y}\left|\frac{f_j(x) - f_j(y)}{x - y}\right| = F_j, \quad \sup_{x\neq y}\left|\frac{g_j(x) - g_j(y)}{x - y}\right| = G_j$$

for $x, y \in \mathbb{R}$, $j = \overline{1,m}$, where F_j and G_j denote positive constants.

$\mathbf{A}_4$. $\|\mathcal{E}\| < 1$, where $\|\cdot\|$ is the spectral matrix norm and $E - |\mathcal{E}|$ is an M-matrix.

The "stability condition" $\|\mathcal{E}\| < 1$ guarantees the existence and uniqueness of the solution of the Cauchy problem. Since $E - |\mathcal{E}|$ is an M-matrix, it is nonsingular and its inverse has only nonnegative entries.

Under these assumptions, for the given initial conditions, there exists a unique solution of system (1). This solution has the form of a vector $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ in which $x_i(t)$ are continuously differentiable for $t \in (0, \beta)$, where β is a positive number, possibly equal to ∞. An equilibrium point of system (1) is denoted by $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$, where the components x_i^* are governed by the algebraic system

$$b_i(x_i^*) = \sum_{j=1}^{m} c_{ij} f_j(x_j^*) + \sum_{j=1}^{m} d_{ij} g_j(x_j^*) + I_i, \quad i = \overline{1, m}. \tag{2}$$

Definition 2. *The equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of system (1) is said to be globally asymptotically stable if any other solution $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ of system (1) is defined for all $t > 0$ and satisfies the equality*

$$\lim_{t\to\infty} x(t) = x^*.$$

3. Existence and Global Asymptotic Stability of Equilibrium Point

Sufficient conditions for the existence and uniqueness of the solution x^* of the algebraic system (2) are given by the following theorem:

Theorem 1 ([11], Theorem 4.2). *Let the assumptions $\mathbf{A}_2$, $\mathbf{A}_3$ be true. In addition, suppose that the following inequalities are satisfied:*

$$\underline{b}_i - \frac{1}{2}\sum_{j=1}^{m} \big(|c_{ij}| F_j + |c_{ji}| F_i\big) - \frac{1}{2}\sum_{j=1}^{m} \big(|d_{ij}| G_j + |d_{ji}| G_i\big) > 0, \quad i = \overline{1, m}. \tag{3}$$

Then system (1) has a unique equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$.

Further, we present sufficient conditions for the global asymptotic stability of the equilibrium point x^* of system (1).

Theorem 2. *Let the assumptions $\mathbf{A}_1 - \mathbf{A}_4$ be true. In addition, suppose that the inequalities*

$$\begin{aligned} &\underline{a}_i \underline{b}_i - \frac{1}{2}\sum_{j=1}^{m} \big(\overline{a}_i |c_{ij}| F_j + \overline{a}_j |c_{ji}| F_i\big) - \frac{1}{2}\sum_{j=1}^{m} \big(\overline{a}_i |d_{ij}| G_j + \overline{a}_j |d_{ji}| G_i\big) \\ &\quad - \frac{1}{2}\sum_{j=1}^{m} \Big(\overline{a}_i \overline{b}_i |e_{ij}| + \overline{a}_j \overline{b}_j |e_{ji}|\Big) - \frac{1}{2}\sum_{j=1}^{m} \overline{a}_j \sum_{k=1}^{m} \big(|c_{ji}|\,|e_{jk}|\,F_i + |c_{jk}|\,|e_{ji}|\,F_k\big) \\ &\quad - \frac{1}{2}\sum_{j=1}^{m} \overline{a}_j \sum_{k=1}^{m} \big(|d_{ji}|\,|e_{jk}|\,G_i + |d_{jk}|\,|e_{ji}|\,G_k\big) > 0, \quad i = \overline{1, m}, \end{aligned} \tag{4}$$

are satisfied and system (1) possesses an equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ *whose components satisfy (2). Then the equilibrium point* x^* *is globally asymptotically stable.*

Remark 1. Inequalities (3) can be deduced from (4) for $\underline{a}_i = \overline{a}_i = 1$ and $e_{ij} = 0$ for $i, j = \overline{1, m}$. However, in general, inequalities (4) do not imply (3).

Remark 2. Inequalities (4) were presented in [11] (Theorem 4.3) as a part of sufficient conditions for the global asymptotic stability of the equilibrium point of the discrete-time counterpart of system (1) provided with impulsive conditions for small values of the discretization step h.

Remark 3. In [8], it is assumed that $g_j = f_j$, the functions $b_i(x_i)$ and $b_i^{-1}(x_i)$ are continuously differentiable, and the functions $b_i'(x_i)$ are bounded both below and above by positive constants. Instead of the m inequalities (4), a single inequality is presented. In our notation can be rewritten as

$$\min_{i=\overline{1,m}} (\underline{a}_i \underline{b}_i) - \max_{i=\overline{1,m}} (\overline{a}_i \overline{b}_i) \|\mathcal{E}\| - \max_{i=\overline{1,m}} \overline{a}_i \left(\max_{i=\overline{1,m}} F_i \|C\| + \max_{i=\overline{1,m}} G_i \|D\| \right) (1 + \|\mathcal{E}\|) > 0, \tag{5}$$

where C and D are $(m \times m)$-matrices with entries c_{ij} and d_{ij}, respectively.

Although condition (5) seems to be much simpler than (4), in our opinion, it is much less precise because the individual lower and upper bounds, Lipschitz constants, and matrix entries are replaced by their minima or maxima, and the matrix norms. In what follows, we present an example of a system satisfying conditions (4) but not conditions (5).

Proof. Introducing the translations

$$u_i(t) = x_i(t) - x_i^*, \quad \varphi_i(s) = \phi_i(s) - x_i^*$$

we deduce the system

$$\dot{u}_i(t) + \sum_{j=1}^m e_{ij} \dot{u}_j(t - \tau_j) = \tilde{a}_i(u_i(t)) \left[-\tilde{b}_i(u_i(t)) + \sum_{j=1}^m c_{ij} \tilde{f}_j(u_j(t)) \right.$$
$$\left. + \sum_{j=1}^m d_{ij} \tilde{g}_j(u_j(t - \tau_j)) \right], \quad t > t_0 = 0, \tag{6}$$
$$u_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \quad i = \overline{1, m},$$

where

$$\tilde{a}_i(u_i) = a_i(u_i + x_i^*), \quad \tilde{b}_i(u_i) = b_i(u_i + x_i^*) - b_i(x_i^*),$$

$$\tilde{f}_j(u_j) = f_j(u_j + x_j^*) - f_j(x_j^*), \quad \tilde{g}_j(u_j) = g_j(u_j + x_j^*) - g_j(x_j^*).$$

This system inherits the assumptions $\mathbf{A}_1 - \mathbf{A}_4$ made above. It suffices to examine the stability characteristics of the trivial equilibrium point $u^* = 0$ of system (6).

We define a Lyapunov functional $V(t)$ by

$$V(t)=\frac{1}{2}\sum_{i=1}^{m}\left\{\left[u_i(t)+\sum_{j=1}^{m}e_{ij}u_j(t-\tau_j)\right]^2+\omega_i\int\limits_{t-\tau_i}^{t}u_i^2(s)\,ds\right\},$$

where ω_i, $i=\overline{1,m}$, are positive constants determined in what follows. First, we note that the value

$$V(0)=\frac{1}{2}\sum_{i=1}^{m}\left\{\left[\varphi_i(0)+\sum_{j=1}^{m}e_{ij}\varphi_j(-\tau_j)\right]^2+\omega_i\int\limits_{-\tau_i}^{0}\varphi_i^2(s)\,ds\right\}$$

is completely determined from the initial values of the system. Then, we compute the rate of changes in $V(t)$ along the solutions of (6) and successively find

$$\dot V(t)=\sum_{i=1}^{m}\left\{\left[u_i(t)+\sum_{j=1}^{m}e_{ij}u_j(t-\tau_j)\right]\left[\dot u_i(t)+\sum_{j=1}^{m}e_{ij}\dot u_j(t-\tau_j)\right]+\frac{\omega_i}{2}(u_i^2(t)-u_i^2(t-\tau_i))\right\}$$

$$=\sum_{i=1}^{m}\left\{\left[u_i(t)+\sum_{j=1}^{m}e_{ij}u_j(t-\tau_j)\right]\tilde a_i(u_i(t))\left[-\tilde b_i(u_i(t))+\sum_{j=1}^{m}c_{ij}\tilde f_j(u_j(t))\right.\right.$$

$$\left.\left.+\sum_{j=1}^{m}d_{ij}\tilde g_j(u_j(t-\tau_j))\right]+\frac{\omega_i}{2}\left(u_i^2(t)-u_i^2(t-\tau_i)\right)\right\}$$

$$=\sum_{i=1}^{m}\left\{-\tilde a_i(u_i(t))\tilde b_i(u_i(t))u_i(t)+\tilde a_i(u_i(t))u_i(t)\left[\sum_{j=1}^{m}c_{ij}\tilde f_j(u_j(t))+\sum_{j=1}^{m}d_{ij}\tilde g_j(u_j(t-\tau_j))\right]\right.$$

$$+\tilde a_i(u_i(t))\sum_{j=1}^{m}e_{ij}u_j(t-\tau_j)\left[-\tilde b_i(u_i(t))+\sum_{j=1}^{m}c_{ij}\tilde f_j(u_j(t))\right.$$

$$\left.\left.+\sum_{j=1}^{m}d_{ij}\tilde g_j(u_j(t-\tau_j))\right]+\frac{\omega_i}{2}\left(u_i^2(t)-u_i^2(t-\tau_i)\right)\right\}$$

$$\le\sum_{i=1}^{m}\left\{-\underline{a}_i\underline{b}_i u_i^2(t)+\overline{a}_i|u_i(t)|\left[\sum_{j=1}^{m}|c_{ij}|\,F_j\,|u_j(t)|+\sum_{j=1}^{m}|d_{ij}|\,G_j\,|u_j(t-\tau_j)|\right]\right.$$

$$+\overline{a}_i\sum_{j=1}^{m}|e_{ij}|\,|u_j(t-\tau_j)|\left[\overline{b}_i\,|u_i(t)|+\sum_{j=1}^{m}|c_{ij}|\,F_j\,|u_j(t)|+\sum_{j=1}^{m}|d_{ij}|\,G_j\,|u_j(t-\tau_j)|\right]$$

$$+ \frac{\omega_i}{2}\left(u_i^2(t) - u_i^2(t-\tau_i)\right)\Bigg\}$$

$$\le \sum_{i=1}^m \Bigg\{ -\underline{a}_i \underline{b}_i u_i^2(t) + \frac{\overline{a}_i}{2}\sum_{j=1}^m |c_{ij}| F_j \left(u_i^2(t) + u_j^2(t)\right) + \frac{\overline{a}_i}{2}\sum_{j=1}^m |d_{ij}|\, G_j \left(u_i^2(t) + u_j^2(t-\tau_j)\right)$$

$$+ \frac{\overline{a}_i \overline{b}_i}{2}\sum_{j=1}^m |e_{ij}| \left(u_i^2(t) + u_j^2(t-\tau_j)\right) + \frac{\overline{a}_i}{2}\sum_{j=1}^m\sum_{k=1}^m |e_{ij}|\,|c_{ik}|\, F_k \left(u_k^2(t) + u_j^2(t-\tau_j)\right)$$

$$+ \frac{\overline{a}_i}{2}\sum_{j=1}^m\sum_{k=1}^m |e_{ij}|\,|d_{ik}|\, G_k \left(u_j^2(t-\tau_j) + u_k^2(t-\tau_k)\right) + \frac{\omega_i}{2}\left(u_i^2(t) - u_i^2(t-\tau_i)\right)\Bigg\}$$

$$= \sum_{i=1}^m \Bigg\{ -\Bigg[\underline{a}_i \underline{b}_i - \frac{1}{2}\Bigg(\overline{a}_i \sum_{j=1}^m |c_{ij}| F_j + F_i \sum_{j=1}^m |c_{ji}| \overline{a}_j \Bigg) - \frac{a_i}{2}\sum_{j=1}^m |d_{ij}| G_j$$

$$- \frac{\overline{a}_i \overline{b}_i}{2}\sum_{j=1}^m |e_{ij}| - \frac{F_i}{2}\sum_{j=1}^m\sum_{k=1}^m |c_{ki}|\,|e_{kj}| \overline{a}_k - \frac{\omega_i}{2}\Bigg] u_i^2(t)$$

$$+ \frac{1}{2}\Bigg[G_i \sum_{j=1}^m |d_{ji}| \overline{a}_j + \sum_{j=1}^m |e_{ji}| \overline{a}_j \overline{b}_j + \sum_{j=1}^m\sum_{k=1}^m |e_{ji}|\,|c_{jk}| \overline{a}_j F_k$$

$$+ \sum_{j=1}^m\sum_{k=1}^m \left(|e_{ji}|\,|d_{jk}| \overline{a}_j G_k + |e_{kj}|\,|d_{ki}| \overline{a}_k G_i\right) - \omega_i \Bigg] u_i^2(t-\tau_i)\Bigg\}.$$

We now choose

$$\omega_i = G_i \sum_{j=1}^m |d_{ji}| \overline{a}_j + \sum_{j=1}^m |e_{ji}| \overline{a}_j \overline{b}_j + \sum_{j=1}^m\sum_{k=1}^m |e_{ji}|\,|c_{jk}| \overline{a}_j F_k$$

$$+ \sum_{j=1}^m\sum_{k=1}^m \left(|e_{ji}|\,|d_{jk}| \overline{a}_j G_k + |e_{kj}|\,|d_{ki}| \overline{a}_k G_i\right) > 0.$$

After some simplifications we obtain

$$\dot{V}(t) \le -\sum_{i=1}^m \Bigg\{ \underline{a}_i \underline{b}_i - \frac{1}{2}\sum_{j=1}^m \left(\overline{a}_i |c_{ij}| F_j + \overline{a}_j |c_{ji}| F_i\right)$$

$$- \frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i|d_{ij}|G_j + \overline{a}_j|d_{ji}|G_i\right) - \frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i\overline{b}_i|e_{ij}| + \overline{a}_j\overline{b}_j|e_{ji}|\right)$$

$$- \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}\left(|c_{ji}|\,|e_{jk}|\,F_i + |c_{jk}|\,|e_{ji}|\,F_k\right)$$

$$\left. - \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}\left(|d_{ji}|\,|e_{jk}|\,G_i + |d_{jk}|\,|e_{ji}|\,G_k\right)\right\} u_i^2(t).$$

According to inequalities (4), there exists $\mu > 0$ such that

$$\mu = \min_{i=\overline{1,m}}\left\{\underline{a}_i\underline{b}_i - \frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i|c_{ij}|F_j + \overline{a}_j|c_{ji}|F_i\right) - \frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i|d_{ij}|G_j + \overline{a}_j|d_{ji}|G_i\right)\right.$$

$$- \frac{1}{2}\sum_{j=1}^{m}\left(\overline{a}_i\overline{b}_i|e_{ij}| + \overline{a}_j\overline{b}_j|e_{ji}|\right) - \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}\left(|c_{ji}|\,|e_{jk}|\,F_i + |c_{jk}|\,|e_{ji}|\,F_k\right)$$

$$\left. - \frac{1}{2}\sum_{j=1}^{m}\overline{a}_j\sum_{k=1}^{m}\left(|d_{ji}|\,|e_{jk}|\,G_i + |d_{jk}|\,|e_{ji}|\,G_k\right)\right\}.$$

Then

$$\dot{V}(t) \le -\mu\|u(t)\|^2, \quad t > 0, \tag{7}$$

where

$$\|v\| = \left(\sum_{i=1}^{m} v_i^2\right)^{1/2}$$

is the Euclidean norm of the vector $v = (v_1, v_2, \ldots, v_m)^T \in \mathbb{R}^m$. Inequality (7) shows that, for any solution $u(t)$ of system (6), the function $V(t)$ is monotonically decreasing and bounded below by 0. Thus, there exists the limit

$$L = \lim_{t\to\infty} V(t) \ge 0.$$

We integrate inequality (7) from 0 to t. This yields

$$V(t) - V(0) \le -\mu\int_0^t \|u(s)\|^2\, ds$$

for all $t > 0$, i.e.,

$$\int_0^t \|u(s)\|^2\, ds \le (V(0) - V(t))/\mu.$$

The last inequality and $L = \lim_{t\to\infty} V(t) \ge 0$ imply that

$$\int_0^\infty \|u(t)\|^2\, dt < \infty. \tag{8}$$

In what follows, we show that the trivial solution of system (6) is stable and (8) implies that

$$\lim_{t\to\infty} \|u(t)\| = 0$$

and, hence,

$$\lim_{t\to\infty} \|x(t) - x^*\| = 0.$$

This means that the equilibrium point x^* of system (1) is globally asymptotically stable.

We complete the proof by using fragments of the arguments from the proofs of Theorems 1.1, 1.3, and 1.4 in [12] (Chapter 8). In the sequel, for a vector $v = (v_1, v_2, \ldots, v_m)^T \in \mathbb{R}^m$ we also use the norm

$$|v| = \max_{i=\overline{1,m}} |v_i|.$$

First, we show that, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if

$$\left| u_i(t) + \sum_{i=1}^m e_{ij} u_j(t - \tau_{ij}) \right| \le \delta_1 \quad \text{for} \quad t \ge 0, \quad i = \overline{1,m}, \quad \text{and} \quad \sup_{s\in[-\tau,0]} |\varphi(s)| \le \delta_1,$$

then $|u(t)| \le \varepsilon$ for $t \ge 0$.

Let T be an arbitrary positive number. For $0 \le t \le T$, we get

$$\begin{aligned}
|u_i(t)| &\le \left| u_i(t) + \sum_{i=1}^m e_{ij} u_j(t - \tau_{ij}) \right| + \left| \sum_{i=1}^m e_{ij} u_j(t - \tau_{ij}) \right| \\
&\le \delta_1 + \sum_{i=1}^m |e_{ij}|\, |u_j(t - \tau_{ij})| \le \delta_1 + \sum_{i=1}^m |e_{ij}| \sup_{-\tau\le t\le T} |u_j(t)| \\
&\le \delta_1 + \sum_{i=1}^m |e_{ij}| \left(\sup_{0\le t\le T} |u_j(t)| + \sup_{-\tau\le s\le 0} |\varphi_j(s)| \right) \\
&\le \sum_{i=1}^m |e_{ij}| \sup_{0\le t\le T} |u_j(t)| + \delta_1 \left(1 + \sum_{i=1}^m |e_{ij}| \right),
\end{aligned}$$

Thus,

$$\sup_{0\le t\le T}|u_i(t)| \le \sum_{i=1}^{m}|e_{ij}|\sup_{0\le t\le T}|u_j(t)| + \delta_1\left(1+\sum_{i=1}^{m}|e_{ij}|\right)$$

or

$$\sup_{0\le t\le T}|u_i(t)| - \sum_{i=1}^{m}|e_{ij}|\sup_{0\le t\le T}|u_j(t)| \le \delta_1\left(1+\sum_{i=1}^{m}|e_{ij}|\right) \quad \text{for} \quad i=\overline{1,m}.$$

If we introduce the vectors

$$U(T)=\left(\sup_{0\le t\le T}|u_1(t)|, \sup_{0\le t\le T}|u_2(t)|,\ldots, \sup_{0\le t\le T}|u_m(t)|\right)^T$$

and

$$\mathbf{e}=(1,1,\ldots,1)^T,$$

then we can represent the last inequalities in the matrix form as follows:

$$(E-|\mathcal{E}|)U(T)\le\delta_1(E+|\mathcal{E}|)\mathbf{e}$$

in a sense of inequalities between the corresponding components of the vectors. Since, by condition $\mathbf{A}_4$, $E-|\mathcal{E}|$ is an M-matrix, we obtain

$$U(T)\le\delta_1(E-|\mathcal{E}|)^{-1}(E+|\mathcal{E}|)\mathbf{e}.$$

We find

$$\sup_{0\le t\le T}|u(t)| = \sup_{0\le t\le T}\max_{i=\overline{1,m}}|u_i(t)| = \max_{i=\overline{1,m}}\sup_{0\le t\le T}|u_i(t)| = |U(T)| \le \delta_1|(E-|\mathcal{E}|)^{-1}(E+|\mathcal{E}|)\mathbf{e}|.$$

If we now choose $\delta_1>0$ sufficiently small such that

$$\delta_1|(E-|\mathcal{E}|)^{-1}(E+|\mathcal{E}|)\mathbf{e}|<\varepsilon,$$

then $|u(t)|\le\varepsilon$ for $0\le t\le T$, where T was an arbitrary positive number. Hence, $|u(t)|\le\varepsilon$ for $t\ge 0$.

We now show that the zero solution of system (6) is stable, i.e.., for any $\varepsilon>0$, there exists $\delta>0$ such that if

$$\sup_{s\in[-\tau,0]}|\varphi(s)|\le\delta,$$

then $|u(t)|\le\varepsilon$ for $t\ge 0$. For any $t\ge 0$ we get

$$\frac{1}{2}\sum_{i=1}^{m}\left[u_i(t)+\sum_{j=1}^{m}e_{ij}u_j(t-\tau_j)\right]^2 \le V(t)\le V(0)$$

$$= \frac{1}{2}\sum_{i=1}^{m}\left\{\left[\varphi_i(0)+\sum_{j=1}^{m}e_{ij}\varphi_j(-\tau_j)\right]^2+\omega_i\int_{-\tau_i}^{0}\varphi_i^2(s)\,ds\right\}$$

$$\le \frac{\delta^2}{2}\sum_{i=1}^{m}\left\{\left[1+\sum_{j=1}^{m}|e_{ij}|\right]^2+\omega_i\tau_i\right\}.$$

If we choose $\delta \in (0,\delta_1)$ sufficiently small such that

$$\delta^2\sum_{i=1}^{m}\left\{\left[1+\sum_{j=1}^{m}|e_{ij}|\right]^2+\omega_i\tau_i\right\}\le \delta_1^2,$$

then

$$\sum_{i=1}^{m}\left[u_i(t)+\sum_{j=1}^{m}e_{ij}u_j(t-\tau_j)\right]^2\le\delta_1^2.$$

This yields

$$\left|u_i(t)+\sum_{i=1}^{m}e_{ij}u_j(t-\tau_{ij})\right|\le\delta_1 \quad \text{for} \quad t\ge 0, \quad i=\overline{1,m},$$

and, consequently, $|u(t)|\le\varepsilon$ for $t\ge 0$.

In view of the stability of the zero solution of system (6) we can assume that $|u(t)|\le h$ for some positive constant h if $\sup_{s\in[-\tau,0]}|\varphi(s)|\le\delta$. Suppose that $\lim_{t\to\infty}u(t)=0$ is not true. In this case, there exists a number $\nu>0$ and an increasing sequence $\{t_k\}$ such that $t_k\to\infty$ and $|u(t_k)|\ge\nu$ for $k\in\mathbb{N}$. For the sake of brevity, we represent system (6) in the form

$$\frac{d}{dt}\left(u_i(t)+\sum_{j=1}^{m}e_{ij}u_j(t-\tau_j)\right)=\mathcal{F}_i(u(t),u(t-\tau)), \quad i=\overline{1,m}, \quad t>0, \tag{9}$$

where

$$\mathcal{F}_i(u,\bar{u}):=\tilde{a}_i(u_i)\left[-\tilde{b}_i(u_i)+\sum_{j=1}^{m}c_{ij}\tilde{f}_j(u_j)+\sum_{j=1}^{m}d_{ij}\tilde{g}_j(\bar{u}_j)\right], \quad i=\overline{1,m}.$$

We denote

$$C_i=\sup_{|u|,|\bar{u}|\le h}|\mathcal{F}_i|, \quad i=\overline{1,m}.$$

For $t\ge 0$ and $\Delta>0$, we now integrate equation (9) from t to $t+\Delta$ to obtain

$$u_i(t+\Delta) - u_i(t) = -\sum_{j=1}^{m} e_{ij}(u_j(t+\Delta-\tau_j) - u_j(t-\tau_j)) + \int_t^{t+\Delta} \mathcal{F}_i(u(s), u(s-\tau))\,ds.$$

Hence,

$$\begin{aligned}|u_i(t+\Delta) - u_i(t)| &\le \sum_{j=1}^{m} |e_{ij}||u_j(t+\Delta-\tau_j) - u_j(t-\tau_j)| \\ &\quad + \int_t^{t+\Delta} |\mathcal{F}_i(u(s), u(s-\tau))|\,ds \\ &\le \sum_{j=1}^{m} |e_{ij}| \sup_{t\ge -\tau} |u_j(t+\Delta) - u_j(t)| + C_i\Delta \\ &\le \sum_{j=1}^{m} |e_{ij}| \left(\sup_{t\ge 0} |u_j(t+\Delta) - u_j(t)| + \sup_{s\in[-\tau,0]} |u_j(s+\Delta) - u_j(s)|\right) + C_i\Delta.\end{aligned}$$

Thus,

$$\begin{aligned}&\sup_{t\ge 0} |u_i(t+\Delta) - u_i(t)| \\ &\quad \le \sum_{j=1}^{m} |e_{ij}| \left(\sup_{t\ge 0} |u_j(t+\Delta) - u_j(t)| + \sup_{s\in[-\tau,0]} |u_j(s+\Delta) - u_j(s)|\right) + C_i\Delta\end{aligned}$$

or

$$\begin{aligned}&\sup_{t\ge 0} |u_i(t+\Delta) - u_i(t)| - \sum_{j=1}^{m} |e_{ij}| \sup_{t\ge 0} |u_j(t+\Delta) - u_j(t)| \\ &\quad \le \sum_{j=1}^{m} |e_{ij}| \sup_{s\in[-\tau,0]} |u_j(s+\Delta) - u_j(s)| + C_i\Delta, \quad i = \overline{1,m}.\end{aligned}$$

If we introduce the vectors

$$\boldsymbol{\rho}(\Delta) = \left(\sup_{t\ge 0} |u_1(t+\Delta) - u_1(t)|, \sup_{t\ge 0} |u_2(t+\Delta) - u_2(t)|, \ldots, \sup_{t\ge 0} |u_m(t+\Delta) - u_m(t)|\right)^T,$$

$$\boldsymbol{\sigma}(\Delta) = \left(\sup_{s\in[-\tau,0]} |u_1(s+\Delta) - u_1(s)|, \sup_{s\in[-\tau,0]} |u_2(s+\Delta) - u_2(s)|, \ldots, \sup_{s\in[-\tau,0]} |u_m(s+\Delta) - u_m(s)|\right)^T,$$

and

$$\mathbf{C} = (C_1, C_2, \dots, C_m)^T,$$

then we can represent the last inequalities in the matrix form as follows:

$$(E - |\mathcal{E}|)\boldsymbol{\rho}(\Delta) \le |\mathcal{E}|\boldsymbol{\sigma}(\Delta) + \Delta\mathbf{C}.$$

As above, this yields

$$\boldsymbol{\rho}(\Delta) \le (E - |\mathcal{E}|)^{-1}(|\mathcal{E}|\boldsymbol{\sigma}(\Delta) + \Delta\mathbf{C})$$

and

$$\sup_{t\ge 0} |u(t+\Delta) - u(t)| \le |(E - |\mathcal{E}|)^{-1}(|\mathcal{E}|\boldsymbol{\sigma}(\Delta) + \Delta\mathbf{C})|. \tag{10}$$

Let $\eta > 0$ and $\Delta \le \eta$. Since $u(t)$ is uniformly continuous on the interval $[-\tau, \eta]$, the right-hand side of (10) can be made arbitrarily small for sufficiently small values of Δ. Thus, we can choose $\eta > 0$ such that $|u(t+\Delta) - u(t)| \le \nu/2$ for all $t \ge 0$ and $\Delta \in [0, \eta]$. In particular,

$$|u(t_k + \Delta)| \ge |u(t_k)| - |u(t_k + \Delta) - u(t_k)| \ge \nu - \frac{\nu}{2} = \frac{\nu}{2}$$

or

$$|u(t)| \ge \frac{\nu}{2} \quad \text{and} \quad \|u(t)\|^2 \ge \frac{\nu^2}{4} \quad \text{for} \quad t \in [t_k, t_k + \eta], \quad k \in \mathbb{N}.$$

Without loss of generality we can assume that the intervals $[t_k, t_k + \eta]$ are disjoint (otherwise, we choose a subsequence). Then

$$\int_0^\infty \|u(t)\|^2\,dt \ge \sum_{k=1}^\infty \int_{t_k}^{t_k+\eta} \|u(t)\|^2\,dt \ge \sum_{k=1}^\infty \eta\frac{\nu^2}{4} = \infty,$$

which contradicts (8). Thus, the relation

$$\lim_{t\to\infty} u(t) = 0$$

is true and the proof is complete.

4. Example

Consider a system

$$\dot{x}_1(t) + 0.1\dot{x}_1(t-\tau_1) + 0.15\dot{x}_2(t-\tau_2) = (2 + 0.01\sin x_1(t))\,[-2x_1(t) + 0.1\arctan x_1(t)$$
$$+ 0.15\arctan x_2(t) + 0.1\arctan x_1(t-\tau_1) + 0.15\arctan x_2(t-\tau_2) + 1]\,,$$

$$\dot{x}_2(t) - 0.2\dot{x}_1(t-\tau_1) + 0.2\dot{x}_2(t-\tau_2) = (3 - 0.02\sin x_2(t))\,[-3x_2(t) + 0.15\arctan x_1(t)$$
$$- 0.2\arctan x_2(t) + 0.1\arctan x_1(t-\tau_1) - -0.2\arctan x_2(t-\tau_2) + 1]\,, \quad t > 0, \tag{11}$$

with arbitrary delays τ_1 and τ_2 and the initial conditions

$$x_i(s) = \phi(s), \quad i = 1, 2, \quad s \in [-\max\{\tau_1, \tau_2\}, 0].$$

System (11) has the form (1). It satisfies the assumptions $\mathbf{A}_1 - \mathbf{A}_4$ with

$$\underline{a}_1 = 1.99, \quad \overline{a}_1 = 2.01, \quad \underline{a}_2 = 2.98, \quad \overline{a}_2 = 3.02, \quad \underline{b}_1 = \overline{b}_1 = 2, \quad \underline{b}_2 = \overline{b}_2 = 3,$$

$$F_1 = F_2 = G_1 = G_2 = 1, \quad \|\mathcal{E}\| = 0.2863903109,$$

and

$$E - |\mathcal{E}| = \begin{bmatrix} 0.9 & -0.15 \\ -0.2 & 0.8 \end{bmatrix}$$

is an M-matrix.

It is easy to see that system (11) satisfies inequalities (3). In fact, the left-hand sides of these inequalities are equal to 1.525 and 2.325 for $i = 1$ and 2, respectively. Thus, system (11) has a unique equilibrium point x^*. It is possible to show that $x^* = (0.6027869379,\ 0.3353919007)^T$.

Furthermore, system (11) satisfies the assumptions of Theorem 2. In fact, the left-hand sides of inequalities (4) are equal to 0.5415 and 4.449 for $i = 1$ and 2, respectively. Therefore, the equilibrium point x^* of system (11) is globally asymptotically stable.

On the other hand, system (11) does not satisfy condition (5) since the left-hand side of the inequality is equal to -0.608775797.

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